

Approximate Analytical Solutions of the Effective Mass Dirac Equation for the generalized Hulthén Potential with any κ -Value

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Abstract

The Dirac equation, with position-dependent mass, is solved approximately for the generalized Hulthén potential with any spin-orbit quantum number κ . Solutions are obtained by using an appropriate coordinate transformation, reducing the effective mass Dirac equation to a Schrödinger-like differential equation. The Nikiforov-Uvarov method is used in the calculations to obtain energy eigenvalues and the corresponding wave functions. Numerical results are compared with those given in the literature. Analytical results are also obtained for the case of constant mass and the results are in good agreement with the literature.

Keywords: Generalized Hulthén potential, Dirac equation, Position-Dependent Mass, Nikiforov-Uvarov Method

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I. INTRODUCTION

In recent years, following work done by von Roos, and Levy-Leblond [1, 2] on relativistic and non-relativistic motion, the position-dependent mass (PDM) formalism has received attention in quantum physics. The solution of the Schrödinger equation with effective mass is a useful starting-point for the investigation of some physical systems, such as the electric properties of quantum wells and quantum dots [3], impurities in crystals [4-6], and semiconductor heterostructures [7]. In general, for both relativistic and non-relativistic cases, the energy spectra and corresponding eigenfunctions have been studied by many authors using different methods and potentials: deformed algebras applied to a Coulomb problem [8], analysis within supersymmetric quantum mechanics [9-11], point canonical transformation to study different classes of potentials [12], non-relativistic Green's functions applied to the harmonic oscillator [13], the Dirac equation in the Coulomb field [14], hyperbolic-type potentials [15], the Morse potential [16], and the Scarf II potential [17].

Another interesting area that has received a lot of attention is the investigation of solutions to the Dirac equation when the Dirac Hamiltonian has either spin or pseudospin symmetry [18]. The Dirac Hamiltonian of a particle with mass μ moving in scalar, $V_s(r)$, and vector, $V_v(r)$, potentials is invariant for two cases: the sum or the difference of the potentials is nearly zero. Important features of these symmetries provide an understanding of the structure of deformed nuclei, including deformation and superdeformation [19, 20], and enable construction of an effective shell-model coupling scheme [21-28].

Recently, pseudospin and spin symmetry have been studied for various potentials, such as the Morse potential [29-31], the Woods-Saxon potential [32], the Coulomb potential [33], and the harmonic potential [34-36]. Some authors have also solved the Dirac equation in the context of pseudospin symmetry under the effect of the Eckart potential [37, 38], and the Pöschl-Teller potential [39, 40], for spin-orbit quantum number $\kappa = 1$ and/or any κ -value. In Ref. [41], the bound states of both the Klein-Gordon and Dirac equations for the Hulthén potential are studied by writing these equations as a Riemann-type equation. In Ref. [42], the energy spectra of the Dirac equation for equal scalar and vector parts of the Hulthén potential is studied using a perturbative approach. In addition, the Dirac equation is solved for the Hulthén potential, for both spin and pseudospin cases, using the asymptotic iteration method [43].

In the present work, we solve the effective mass Dirac equation for the generalized Hulthén potential within the framework of an approximation to the $\kappa(\kappa - 1)/r^2$ term. We give the energy eigenvalue equation and the corresponding eigenfunctions for any spin-orbit quantum number κ . We also obtain the energy eigenvalue equation in the case of constant mass. We give separately the Dirac solutions for the cases where spin and pseudospin symmetries are taken into account. We apply the parametric generalization of the Nikiforov-Uvarov (NU) method to obtain the energy eigenvalue equations and the eigenfunctions of the generalized Hulthén potential. The NU method is a powerful method for solving second order differential equations [44], like methods based on Lie algebras [45]. In Ref. [46], this approach was generalized to a theory based on spectrum generating algebras including relativistic effects. The NU method can be used, for both non-relativistic and relativistic cases, as a tool to find the energy spectrum and corresponding wave functions.

II. DIRAC EQUATION

The Dirac equation for a spin- $\frac{1}{2}$ particle with mass μ moving in scalar $V_s(r)$, and vector $V_v(r)$ potentials is written as

$$[\alpha \cdot \mathbf{P} + \beta[\mu + V_s(r)] + V_v(r) - E]\Psi(r) = 0, \quad (1)$$

where E is the relativistic energy of the particle, \mathbf{P} is three-momentum, and α and β are 4×4 Dirac matrices [47] defined, respectively as,

$$\alpha = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \quad (2)$$

Here, σ is a three-vector whose components are Pauli matrices, and I denotes the 2×2 unit matrix. For a spherically symmetric potential, the total angular momentum \mathbf{J} , and spin-orbit operator $\hat{K} = -\beta(\sigma \cdot \mathbf{L} + 1)$ commute with the Dirac Hamiltonian, where \mathbf{L} is the orbital angular momentum operator. The eigenvalues of the operator \hat{K} are $\kappa = \pm(j + 1/2)$; $\kappa = -(j + 1/2) < 0$ stands for the aligned spin $j = \ell + 1/2$, and $\kappa = (j + 1/2) > 0$ stands for the unaligned spin $j = \ell - 1/2$. On the other hand, the total angular quantum number can be written in terms of the pseudo-orbital angular momentum $\tilde{\ell} = \ell + 1$ and the

pseudospin angular momentum $\tilde{s} = 1/2$ as $j = \tilde{\ell} + \tilde{s}$. The spin-orbit quantum number κ is related to the orbital angular quantum number by the expressions $\kappa(\kappa + 1) = \ell(\ell + 1)$, and $\kappa(\kappa - 1) = \tilde{\ell}(\tilde{\ell} + 1)$ for a given $\kappa = \pm 1, \pm 2, \dots$. The spherically symmetric Dirac spinor can be written in terms of upper and lower components,

$$\Psi_{n\kappa}(r) = \frac{1}{r} \begin{pmatrix} F_{n\kappa}(r) Y_{jm}^{\ell}(\theta, \phi) \\ iG_{n\kappa}(r) Y_{jm}^{\tilde{\ell}}(\theta, \phi) \end{pmatrix}, \quad (3)$$

where $Y_{jm}^{\ell}(\theta, \phi)$, and $Y_{jm}^{\tilde{\ell}}(\theta, \phi)$ are spherical harmonics, and $F_{n\kappa}(r)/r$, and $G_{n\kappa}(r)/r$ are the radial parts of the upper and lower components. Substituting Eq. (3) into Eq. (1) one can write the Dirac equation as a set of two coupled differential equations in terms of $F_{n\kappa}(r)$ and $G_{n\kappa}(r)$

$$\left(\frac{d}{dr} + \frac{\kappa}{r} \right) F_{n\kappa}(r) = [\mu + E_{n\kappa} - V_-(r)] G_{n\kappa}(r), \quad (4)$$

$$\left(\frac{d}{dr} - \frac{\kappa}{r} \right) G_{n\kappa}(r) = [\mu - E_{n\kappa} + V_+(r)] F_{n\kappa}(r), \quad (5)$$

where $V_-(r) = V_v(r) - V_s(r)$, and $V_+(r) = V_v(r) + V_s(r)$. Using Eq. (4) for the upper component, and substituting into Eq. (5), we obtain generalized differential equations for the position-dependent mass case,

$$\left(\frac{d^2}{dr^2} - \frac{\kappa(\kappa + 1)}{r^2} - [\mu + E_{n\kappa} - V_-(r)][\mu - E_{n\kappa} + V_+(r)] \right. \\ \left. - \frac{(\frac{d\mu(r)}{dr} - \frac{dV_-(r)}{dr})(\frac{d}{dr} + \frac{\kappa}{r})}{\mu + E_{n\kappa} - V_-(r)} \right) F_{n\kappa}(r) = 0, \quad (6)$$

$$\left(\frac{d^2}{dr^2} - \frac{\kappa(\kappa - 1)}{r^2} - [\mu + E_{n\kappa} - V_-(r)][\mu - E_{n\kappa} + V_+(r)] \right. \\ \left. - \frac{(\frac{d\mu(r)}{dr} + \frac{dV_+(r)}{dr})(\frac{d}{dr} - \frac{\kappa}{r})}{\mu - E_{n\kappa} + V_+(r)} \right) G_{n\kappa}(r) = 0. \quad (7)$$

Here, the energy eigenvalues depend on the quantum numbers n , and κ , and also on the quantum number $\tilde{\ell}$ according to the relation $\kappa(\kappa - 1) = \tilde{\ell}(\tilde{\ell} + 1)$. To solve these equations, we approximate the centrifugal term. Thus, the energy spectra and the corresponding eigenfunctions can be obtained analytically by using the NU method.

III. PARAMETRIC FORMULATION OF NIKIFOROV-UVAROV METHOD

We briefly give the mathematical background required for the parametric NU method [48], where the general form of the Schrödinger-like equation for any potential is written as

$$\left[\frac{d^2}{ds^2} + \frac{\alpha_1 - \alpha_2 s}{s(1 - \alpha_3 s)} \frac{d}{ds} + \frac{-\xi_1 s^2 + \xi_2 s - \xi_3}{[s(1 - \alpha_3 s)]^2} \right] \psi(s) = 0. \quad (8)$$

Comparing Eq. (8) with the general form of the basic equation of the method, we obtain

$$\tilde{\tau}(s) = \alpha_1 - \alpha_2 s \quad ; \quad \sigma(s) = s(1 - \alpha_3 s) \quad ; \quad \tilde{\sigma}(s) = -\xi_1 s^2 + \xi_2 s - \xi_3. \quad (9)$$

Substituting Eq. (9) into the polynomial $\pi(s) = \frac{(\sigma' - \tilde{\tau})}{2} \pm \sqrt{\left(\frac{\sigma' - \tilde{\tau}}{2}\right)^2 - \tilde{\sigma} + k\sigma}$, we obtain

$$\pi(s) = \alpha_4 + \alpha_5 s \pm \sqrt{(\alpha_6 - k\alpha_3)s^2 + (\alpha_7 + k)s + \alpha_8}, \quad (10)$$

where $\alpha_4 = \frac{1}{2}(1 - \alpha_1)$, $\alpha_5 = \frac{1}{2}(\alpha_2 - 2\alpha_3)$, $\alpha_6 = \alpha_5^2 + \xi_1$, $\alpha_7 = 2\alpha_4\alpha_5 - \xi_2$, and $\alpha_8 = \alpha_4^2 + \xi_3$.

In the NU-method, the function under the square root in Eq. (10) must be the square of a polynomial [48]. This condition gives the roots of the parameter k ,

$$k_{1,2} = -(\alpha_7 + 2\alpha_3\alpha_8) \pm 2\sqrt{\alpha_8\alpha_9}, \quad (11)$$

with $\alpha_9 = \alpha_3\alpha_7 + \alpha_3^2\alpha_8 + \alpha_6$. For $k = -(\alpha_7 + 2\alpha_3\alpha_8) - 2\sqrt{\alpha_8\alpha_9}$, $\pi(s)$ becomes

$$\pi(s) = \alpha_4 + \alpha_5 s - [(\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8})s - \sqrt{\alpha_8}], \quad (12)$$

and also

$$\tau(s) = \alpha_1 + 2\alpha_4 - (\alpha_2 - 2\alpha_5)s - 2[(\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8})s - \sqrt{\alpha_8}]. \quad (13)$$

To satisfy the condition that the derivative of the function $\tau(s)$ should be negative, we impose

$$\begin{aligned}
\tau'(s) &= -(\alpha_2 - 2\alpha_5) - 2(\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8}) \\
&= -2\alpha_3 - 2(\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8}) < 0.
\end{aligned} \tag{14}$$

The energy eigenvalue equation is written as [48]

$$\begin{aligned}
\alpha_2 n - (2n+1)\alpha_5 + (2n+1)(\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8}) + n(n-1)\alpha_3 \\
+ \alpha_7 + 2\alpha_3\alpha_8 + 2\sqrt{\alpha_8\alpha_9} = 0.
\end{aligned} \tag{15}$$

and the equality $\sigma'(s)\rho(s) + \sigma(s)\rho'(s) = \tau(s)\rho(s)$ gives

$$\rho(s) = s^{\alpha_{10}-1} (1 - \alpha_3 s)^{\frac{\alpha_{11}}{\alpha_3} - \alpha_{10} - 1}. \tag{16}$$

This equation together with $y_n(s) \sim \frac{1}{\rho(s)} \frac{d^n}{ds^n} [\sigma^n(s) \rho(s)]$ gives

$$y_n(s) = P_n^{(\alpha_{10}-1, \frac{\alpha_{11}}{\alpha_3} - \alpha_{10} - 1)} (1 - 2\alpha_3 s), \tag{17}$$

where $\alpha_{10} = \alpha_1 + 2\alpha_4 + 2\sqrt{\alpha_8}$, $\alpha_{11} = \alpha_2 - 2\alpha_5 + 2(\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8})$ and $P_n^{(\alpha, \beta)}(1 - 2\alpha_3 s)$ are the Jacobi polynomials. The equality $\frac{\phi'(s)}{\phi(s)} = \frac{\pi(s)}{\sigma(s)}$ [48] gives

$$\phi(s) = s^{\alpha_{12}} (1 - \alpha_3 s)^{-\alpha_{12} - \frac{\alpha_{13}}{\alpha_3}}, \tag{18}$$

and the general solution $\psi(s) = \phi(s)y(s)$ becomes

$$\psi(s) = s^{\alpha_{12}} (1 - \alpha_3 s)^{-\alpha_{12} - \frac{\alpha_{13}}{\alpha_3}} P_n^{(\alpha_{10}-1, \frac{\alpha_{11}}{\alpha_3} - \alpha_{10} - 1)} (1 - 2\alpha_3 s), \tag{19}$$

where $\alpha_{12} = \alpha_4 + \sqrt{\alpha_8}$ and $\alpha_{13} = \alpha_5 - (\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8})$ [48].

IV. BOUND-STATE SOLUTIONS

In order to solve the Dirac equation, including the term proportional to $1/r^2$, we set the vector and scalar potentials as the generalized Hulthén potential [31, 49] of the forms

$$V_v(r) = V_0 \frac{e^{-2\alpha r}}{e^{-2\alpha r} - 1}, V_0 > 0, \quad (20)$$

$$V_s(r) = -S_0 \frac{e^{-2\alpha r}}{e^{-2\alpha r} - 1}, S_0 > 0, \quad (21)$$

where α is the screening parameter (positive), the constant parameter S_0 denotes the scalar and V_0 denotes the vector part of the potential, respectively. Using the last two equations, we obtain

$$V_-(r) = (V_0 + S_0) \frac{e^{-2\alpha r}}{e^{-2\alpha r} - 1}, \quad (22)$$

$$V_+(r) = (V_0 - S_0) \frac{e^{-2\alpha r}}{e^{-2\alpha r} - 1}. \quad (23)$$

where the “effective” potential $V_-(r)$ is a repulsive one in the relativistic region [18, 26-28, 35, 50, 51]. It is well known that the Hulthén potential gives bound states in the non-relativistic region only if the potential parameters satisfy the condition that $\delta^2 < V_0 < 4\delta^2$, where $\delta^2 = 4\alpha^2$, in the absence of the scalar part [41].

We use the equality $d\mu(r)/dr = -dV_+(r)/dr$ to eliminate the last term in Eq. (7), and obtain the mass function which can be written as

$$\mu(r) = \mu_0 - \frac{\mu_1}{1 - e^{2\alpha r}}. \quad (24)$$

The mass function has the same form as the Hulthén potential, where μ_0 denotes the integral constant, and $\mu_1 = V_0 - S_0$. Thus, the parameter μ_1 contains contributions coming from the scalar, as well as the vector part of the potential. The parameter μ_0 corresponds to the rest mass of the Dirac particle. Substituting Eqs. (22), (23) and (24) into Eq. (7), we get

$$\left\{ \frac{d^2}{dr^2} - \frac{\kappa(\kappa - 1)}{r^2} - \left(\mu_0 - \mu_1 \frac{e^{-2\alpha r}}{e^{-2\alpha r} - 1} - (V_0 + S_0) \frac{e^{-2\alpha r}}{e^{-2\alpha r} - 1} + E_{nk} \right) \right. \\ \left. \times \left(\mu_0 - \mu_1 \frac{e^{-2\alpha r}}{e^{-2\alpha r} - 1} + (V_0 - S_0) \frac{e^{-2\alpha r}}{e^{-2\alpha r} - 1} - E_{nk} \right) \right\} G_{n\kappa}(r) = 0, \quad (25)$$

The following approximation is used for $1/r^2$ term [31, 52]

$$\frac{1}{r^2} \approx \frac{4\alpha^2 e^{-2\alpha r}}{(1 - e^{-2\alpha r})^2}, \quad (26)$$

We compute $1/r^2$ expanding into series. This will provide a physical result for $\alpha = 0$.

Defining a new variable $s = e^{-2\alpha r}$, we have

$$\left\{ \frac{d^2}{ds^2} + \frac{1-s}{s(1-s)} \frac{d}{ds} + \frac{1}{[s(1-s)]^2} [\eta^2(E_{nk}^2 - \mu_0^2) - 2\eta^2[\mu_0(S_0 - \mu_0) - E_{n\kappa}V_0 + E_{nk}^2 + 2\alpha^2\kappa(\kappa - 1) + \mu_1\mu_0]s - \eta^2[\mu_0^2 - E_{nk}^2 + 2E_{n\kappa}V_0 - 2S_0\mu_0 + \mu_1(\mu_1 + 2S_0 - 2\mu_0) - V_0^2 + S_0^2]s^2] \right\} G_{n\kappa}(s) = 0, \quad (27)$$

where $\eta^2 = 1/4\alpha^2$. Comparing Eq. (27) with Eq. (8), we get the parameter set given in Section II:

$$\begin{aligned} \alpha_1 &= 1, & -\xi_1 &= \eta^2[\mu_0^2 - E_{nk}^2 + 2E_{n\kappa}V_0 - 2S_0\mu_0 + \mu_1(\mu_1 + 2S_0 - 2\mu_0) - V_0^2 + S_0^2] \\ \alpha_2 &= 1, & \xi_2 &= -2\eta^2[\mu_0(S_0 - \mu_0) - E_{n\kappa}V_0 + E_{nk}^2 + 2\alpha^2\kappa(\kappa - 1) + \mu_1\mu_0] \\ \alpha_3 &= 1, & -\xi_3 &= \eta^2(E_{nk}^2 - \mu_0^2) \\ \alpha_4 &= 0, & \alpha_5 &= -\frac{1}{2} \\ \alpha_6 &= \xi_1 + \frac{1}{4}, & \alpha_7 &= -\xi_2 \\ \alpha_8 &= \xi_3, & \alpha_9 &= \xi_1 - \xi_2 + \xi_3 + \frac{1}{4} \\ \alpha_{10} &= 1 + 2\sqrt{\xi_3}, & \alpha_{11} &= 2 + 2(\sqrt{\xi_1 - \xi_2 + \xi_3 + \frac{1}{4}} + \sqrt{\xi_3}) \\ \alpha_{12} &= \sqrt{\xi_3}, & \alpha_{13} &= -\frac{1}{2} - (\sqrt{\xi_1 - \xi_2 + \xi_3 + \frac{1}{4}} + \sqrt{\xi_3}) \end{aligned} \quad (28)$$

We can easily obtain the energy eigenvalue equation of the generalized Hulthén potential for any κ value from Eq. (15)

$$2\eta\sqrt{\mu_0^2 - E_{n\kappa}^2} = \frac{\eta^2}{N} (2E_{n\kappa}V_0 + \mu_1(\mu_1 + 2S_0 - 2\mu_0) - V_0^2 + S_0^2 - 2\mu_0S_0) - N, \quad (29)$$

where

$$N = \frac{1}{2}(2n + 1) + \sqrt{\eta^2[\mu_1(\mu_1 + 2S_0) - V_0^2 + S_0^2 + 4\alpha^2\kappa(\kappa - 1)] + \frac{1}{4}}. \quad (30)$$

The numerical results for different quantum numbers (n, κ) are listed in Table I. We list the eigenvalues $E_{n\kappa}$ for constant mass ($\mu_1 = 0$) and for two different values of μ_1 to see the

effect of the spatially dependent mass. The results given for constant mass are compared with results reported in the literature. For the constant mass case, we see good agreement with the results given in Ref. [42], where the relativistic energy W is stated as $W = E + m$.

The lower spinor component can be obtained from Eq. (19)

$$G_{n\kappa}(s) = s^{\epsilon_{n\kappa}}(1-s)^{1/2+\delta}P_n^{(2\epsilon, 2\delta)}(1-2s), \quad (31)$$

where $\epsilon_{n\kappa} = \sqrt{\eta^2(\mu_0 - E_\kappa^2)}$, and $\delta = \sqrt{\eta^2[\mu_1(\mu_1 + 2S_0) - V_0^2 + S_0^2 + 4\alpha^2\kappa(\kappa - 1)] + \frac{1}{4}}$.

From the last equation, and using Eq. (5), the upper spinor component can be expressed as

$$\begin{aligned} F_{n\kappa}(s) = & \frac{2\alpha s^{\epsilon_{n\kappa}}(1-s)^{\frac{1}{2}+\delta}}{\mu(s) - E_{n\kappa} + \Sigma(s)} \left\{ \left[\frac{s}{1-s} \left(\frac{1}{2} + \delta \right) - \epsilon_{n\kappa} + \frac{\kappa}{lns} \right] P_n^{(2\epsilon_{n\kappa}, 2\delta)}(1-2s) \right. \\ & \left. - \frac{1}{2}(n + 2\epsilon_{n\kappa} + 2\delta + 1)P_{n-1}^{(1+2\epsilon_{n\kappa}, 1+2\delta)}(1-2s) \right\}. \end{aligned} \quad (32)$$

We consider the case of constant mass to discuss the compatibility of our results. Setting $\mu_1 = 0$ in Eq. (24) while keeping in mind that $V_+(r) = C = \text{const.}$ for pseudospin symmetry, and following the same procedure ($s = e^{-2\alpha r}$), we obtain the energy eigenvalue equation from Eq. (7)

$$\begin{aligned} (\mu_0 + E_{n\kappa})(\mu_0 - E_{n\kappa} + C) = & \frac{1}{4\eta^2} \left[\frac{1}{2}(2n+1) + \sqrt{\kappa(\kappa-1) + \frac{1}{4}} \right. \\ & \left. + \eta^2 V \frac{\mu_0 - E_{n\kappa} + C}{\frac{1}{2}(2n+1) + \sqrt{\kappa(\kappa-1) + \frac{1}{4}}} \right]^2, \end{aligned} \quad (33)$$

where $V = V_0 + S_0$. The result in the case of constant mass is Eq. (47) of Ref. [43] for $S_0 \rightarrow 0$ under the exact pseudospin symmetry.

We now briefly give the bound state solutions for the constant mass case under spin symmetry. Setting $V_-(r) = C = \text{const.}$, and $\mu_1 = 0$, we get the bound state solutions under the exact spin symmetry for the case of constant mass from Eq. (6) as ($s = e^{-2\alpha r}$)

$$\begin{aligned} (\mu_0 - E_{n\kappa})(\mu_0 + E_{n\kappa} - C) = & \frac{1}{4\eta'^2} \left[\frac{1}{2}(2n+1) + \sqrt{\kappa(\kappa+1) + \frac{1}{4}} \right. \\ & \left. + \eta'^2 V' \frac{\mu_0 + E_{n\kappa} - C}{\frac{1}{2}(2n+1) + \sqrt{\kappa(\kappa+1) + \frac{1}{4}}} \right]^2, \end{aligned} \quad (34)$$

where $V' = S_0 - V_0$.

V. CONCLUSION

We approximately solved the Dirac equation, with position-dependent mass, for the generalized Hulthén potential with arbitrary spin-orbit quantum number. We found the eigenvalue equation, and corresponding two-component spinors in terms of Jacobi polynomials by using the parametric generalization of the NU-method within the framework of an approximation to the $\kappa(\kappa - 1)/r^2$ term. We compared the numerical results with those obtained in the literature and given in Table I. We showed results for the case of constant mass, and summarized the results for two different position-dependent mass values, obtained when $\mu_1 = 0.005$ and $\mu_1 = 0.0001$. We also obtained the energy eigenvalue equation for the constant mass case with spin and pseudospin symmetries, separately. These analytical results are in agreement with results reported in the literature.

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TABLE I: Energy eigenvalues for different values of n and κ ($\mu_0 = V_0 = 1$).

μ_1	α	$n \ \kappa$	state	$E_{n\kappa} < 0$	Ref.[47]
0	0.025	1 -1	$1s_{1/2}$	0.998068	-1.993900
	0.01			0.999691	-1.999000
	0.05			0.992258	-1.975800
	0.1			0.968772	-1.905700
0.005	0.025	1 -1	$1s_{1/2}$	—	—
		1 -2	$1p_{3/2}$	0.998068	—
		1 -3	$1d_{5/2}$	0.995895	—
		1 -4	$1f_{7/2}$	0.993424	—
	0.05	1 -1	$1s_{1/2}$	0.994106	—
		1 -2	$1p_{3/2}$	0.987615	—
		1 -3	$1d_{5/2}$	0.979641	—
		1 -4	$1f_{7/2}$	0.969946	—
	0.1	1 -1	$1s_{1/2}$	0.970550	—
		1 -2	$1p_{3/2}$	0.945421	—
		1 -3	$1d_{5/2}$	0.912747	—
		1 -4	$1f_{7/2}$	0.871793	—
	0.025	1 -1	$1s_{1/2}$	0.998103	—
		1 -2	$1p_{3/2}$	0.996592	—
		1 -3	$1d_{5/2}$	0.994653	—
		1 -4	$1f_{7/2}$	0.992281	—
	0.05	1 -1	$1s_{1/2}$	0.992293	—
		1 -2	$1p_{3/2}$	0.986235	—
		1 -3	$1d_{5/2}$	0.978413	—
		1 -4	$1f_{7/2}$	0.968795	—
	0.1	1 -1	$1s_{1/2}$	0.968807	—
		1 -2	$1p_{3/2}$	0.944015	—
		1 -3	$1d_{5/2}$	0.911464	—
		1 -4	$1f_{7/2}$	0.870558	—